4-6 Voltage and Electric Potential

Reading Assignment: *pp. 107-116*

$$\int_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell}$$

$$\int_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = g(\bar{r}_2) - g(\bar{r}_1), \text{ where } \mathbf{E}(\bar{r}) = \nabla g(\bar{r})$$

Q:

A: HO: Voltage and Electric Potential

1. a static electric field $\mathbf{E}(\bar{r})$ (a **vector** field).

2. an electrostatic potential field $V(\bar{r})$ (a **scalar** field!).

Q:

A:

HO: Electric Potential for Point Charge

Q:

A:

HO: Electric Potential Function for Charge Densities

Example: The Electric Dipole

Q:

A: HO: The Dipole Moment

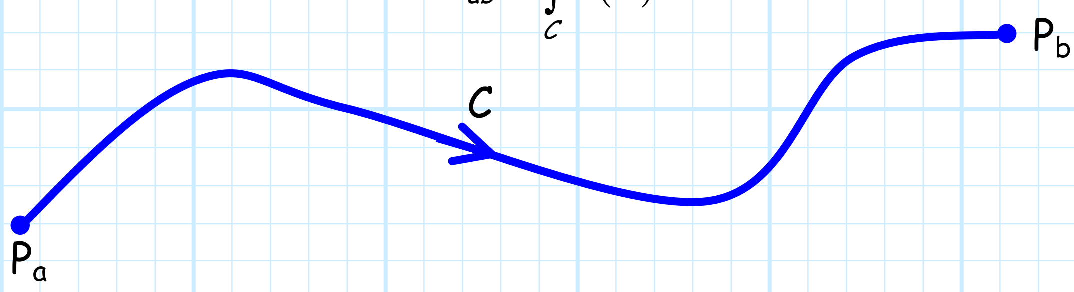
Voltage and Electric Potential

An important application of the line integral is the calculation of **work**. Say there is some vector field $\mathbf{F}(\bar{r})$ that exerts a **force** on some object.

Q: *How much **work** (W) is done by this vector field if the object moves from point P_a to P_b, along contour C ??*

A: We can find out by evaluating the line integral:

$$W_{ab} = \int_C \mathbf{F}(\bar{r}) \cdot \overline{d\ell}$$

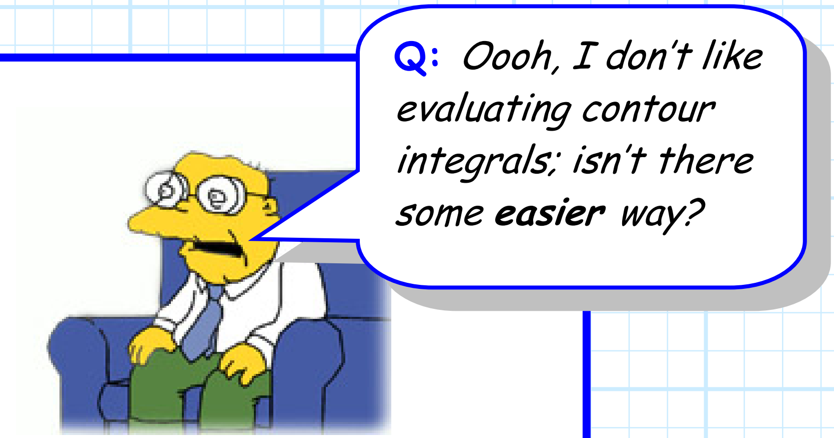


Say this object is a **charged particle** with charge Q, and the force is applied by a static **electric field** $\mathbf{E}(\bar{r})$. We **know** the force on the charged particle is:

$$\mathbf{F}(\bar{r}) = Q\,\mathbf{E}(\bar{r})$$

and thus the work done **by the electric field** in moving a charged particle along some contour C is:

$$W_{ab} = \int_C \mathbf{F}(\bar{\mathrm{r}}) \cdot \overline{d\ell}$$
$$= Q \int_C \mathbf{E}(\bar{\mathrm{r}}) \cdot \overline{d\ell}$$



A: **Yes** there is! Recall that a **static** electric field is a **conservative** vector field. Therefore, we can write any electric field as the **gradient** of a specific **scalar** field $V(\bar{\mathrm{r}})$:

$$\mathbf{E}(\bar{\mathrm{r}}) = -\nabla V(\bar{\mathrm{r}})$$

We can then evaluate the work integral as:

$$\begin{aligned} W_{ab} &= Q \int_C \mathbf{E}(\bar{\mathrm{r}}) \cdot \overline{d\ell} \\ &= -Q \int_C \nabla V(\bar{\mathrm{r}}) \cdot \overline{d\ell} \\ &= -Q \left[V(\bar{\mathrm{r}}_b) - V(\bar{\mathrm{r}}_a) \right] \\ &= Q \left[V(\bar{\mathrm{r}}_a) - V(\bar{\mathrm{r}}_b) \right] \end{aligned}$$

We **define**:

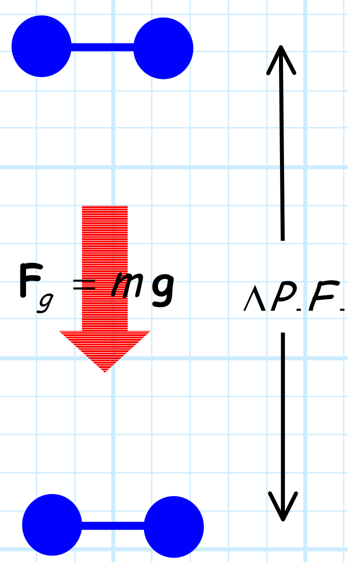
$$V_{ab} \doteq V(\bar{r}_a) - V(\bar{r}_b)$$

Therefore:

$$W_{ab} = Q\, V_{ab}$$

Q: *So what the heck* ***is*** V_{ab} *? Does it mean any thing? Do we use it in engineering?*

A: First, consider what W_{ab} is!



The value W_{ab} represents the work done **by** the electric field **on** charge Q when moving it from point P_a to point P_b. This is **precisely** the same concept as when a **gravitational force** field moves an object from one point to another.

The work done by the gravitational field in this case is equal to the **difference** in **potential energy** (*P.E.*) between the object at these two points.

The value W_{ab} represents the **same** thing! It is the **difference** in **potential energy** between the charge at point P_a and at P_b.

Q: *Great, now we know what W_{ab} is. But the question was, **WHAT IS** V_{ab} !?!*

A: That's easy! Just rearrange the above equation:

$$V_{ab} = \frac{W_{ab}}{Q}$$

See? The value V_{ab} is equal to the difference in potential energy, **per coulomb of charge**!

* In other words V_{ab} represents the difference in potential energy for **each** coulomb of charge in Q.

* Another way to look at it: V_{ab} is the difference in potential energy if the particle has a charge of **1 Coulomb** (i.e., $Q=1$).

Note that V_{ab} can be expressed as:

$$\begin{aligned} V_{ab} &= \int_C \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{d\ell} \\ &= V(\bar{r}_a) - V(\bar{r}_b) \end{aligned}$$

where point P_a lies at the **beginning** of contour C, and P_b lies at the **end**.

We refer to the **scalar field** $V(\bar{r})$ as the **electric potential function**, or the electric potential **field**.

We likewise refer to the scalar value V_{ab} as the electric potential **difference**, or simply the **potential difference** between point P_a and point P_b.

Note that V_{ab} (and therefore $V(\bar{r})$), has units of:

$$V_{ab} = \frac{W_{ab}}{Q} \quad \left[\frac{\text{Joules}}{\text{Coulomb}}\right]$$

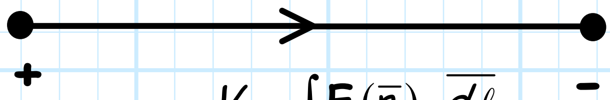
Joules/Coulomb is a rather **awkward** unit, so we will use the other name for it—**VOLTS**!

$$\frac{1 \text{ Joule}}{\text{Coulomb}} \doteq 1 \text{ Volt}$$

Q: *Hey! We used volts in **circuits** class. Is this the **same** thing ?*

A: It is **precisely** the same thing !

Perhaps this will help. Say P_a and P_b are two points somewhere on a circuit. But let's call these points something different, say point + and point - .



$$V = \int_C \mathbf{E}(\bar{\mathbf{r}}) \cdot \overline{d\ell}$$

Therefore, V represents the **potential difference** (in volts) **between** point (i.e., **node**) + and point (**node**) - . Note this value can be either **positive** or **negative**.

Q: *But, does this mean that circuits produce **electric fields**?*

A: **Absolutely**! Anytime you can measure a **voltage** (i.e., a potential difference) between two points, an electric field **must** be present!

Electric Potential for Point Charge

Recall that a point charge Q, **located at the origin** ($\bar{r}'=0$), produces a static electric field:

$$\mathbf{E}(\bar{\mathbf{r}}) = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{a}_r$$

Now, we know that this field is the **gradient** of some scalar field:

$$\mathbf{E}(\bar{\mathbf{r}}) = -\nabla V(\bar{\mathbf{r}})$$

Q: What is the **electric potential** function $V(\bar{\mathbf{r}})$ generated by a **point charge** Q, located at the origin?

A: We find that it is:

$$V(\bar{\mathbf{r}}) = \frac{Q}{4\pi\varepsilon_0 r}$$

Q: *Where did **this** come from ? How do we know that this is the correct solution?*

A: We can show it is the correct solution by **direct substitution**!

$$\begin{aligned}\mathbf{E}(\bar{\mathbf{r}}) &= -\nabla V(\bar{\mathbf{r}}) \\ &= -\nabla\left(\frac{Q}{4\pi\varepsilon_0 r}\right) \\ &= -\frac{\partial}{\partial r}\left(\frac{Q}{4\pi\varepsilon_0 r}\right)\hat{a}_r + 0 \\ &= \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{a}_r\end{aligned}$$

The **correct** result!

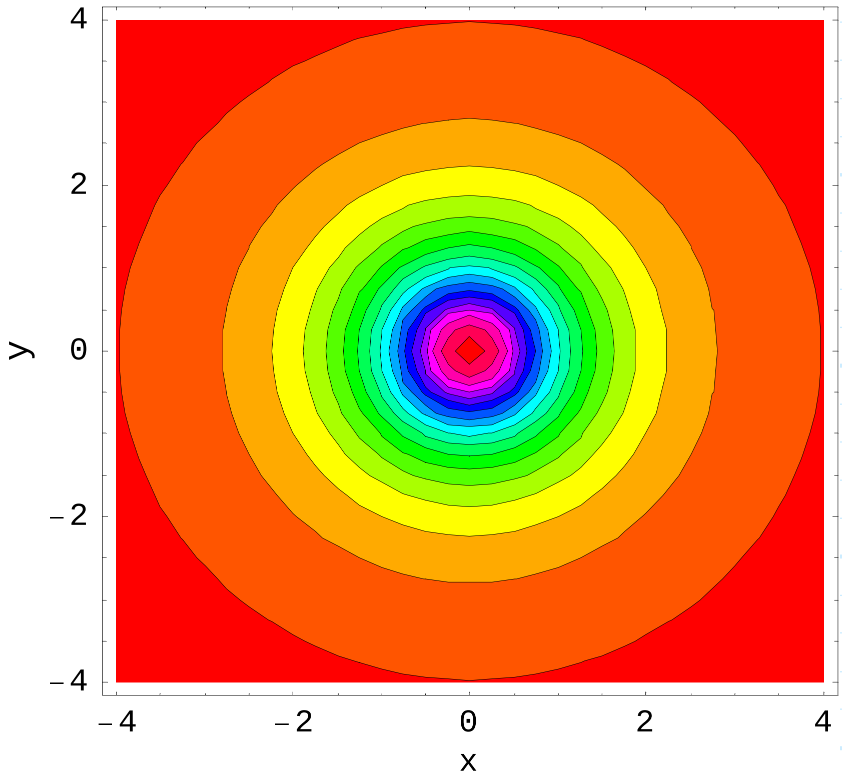
Q: *What if the charge is **not** located at the **origin**?*

A: **Substitute** r with $|\bar{\mathbf{r}}-\bar{\mathbf{r}}'|$, and we get:

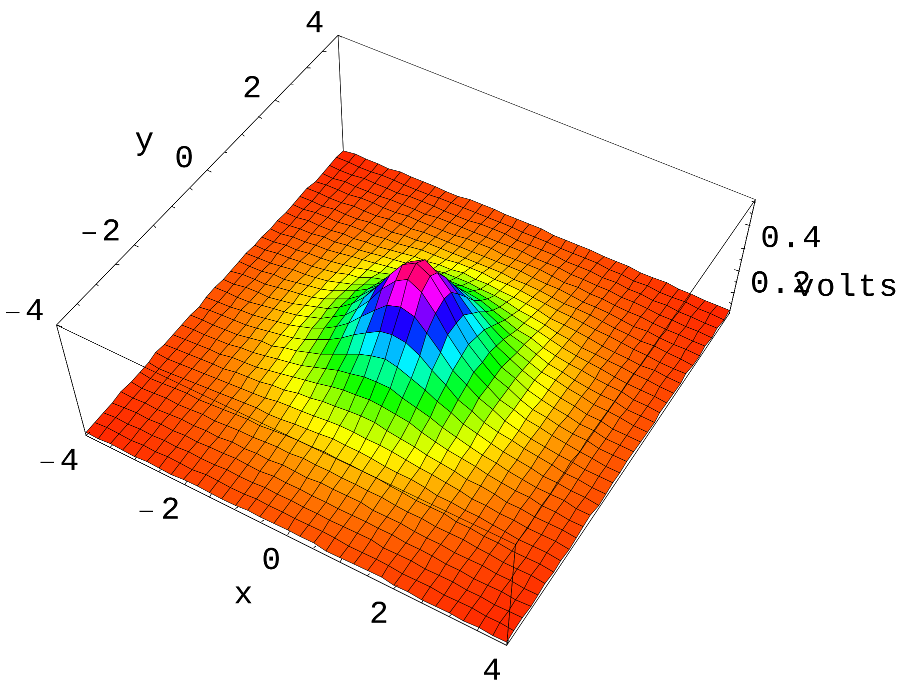
$$V(\bar{\mathbf{r}}) = \frac{Q}{4\pi\varepsilon_0 |\bar{\mathbf{r}}-\bar{\mathbf{r}}'|}$$

Where, as before, the position vector $\bar{\mathbf{r}}'$ denotes the location of the **charge** Q, and the position vector $\bar{\mathbf{r}}$ denotes the location in space where the electric potential function is **evaluated**.

The **scalar** function $V(\bar{r})$ for a point charge can be shown graphically as a **contour plot**:

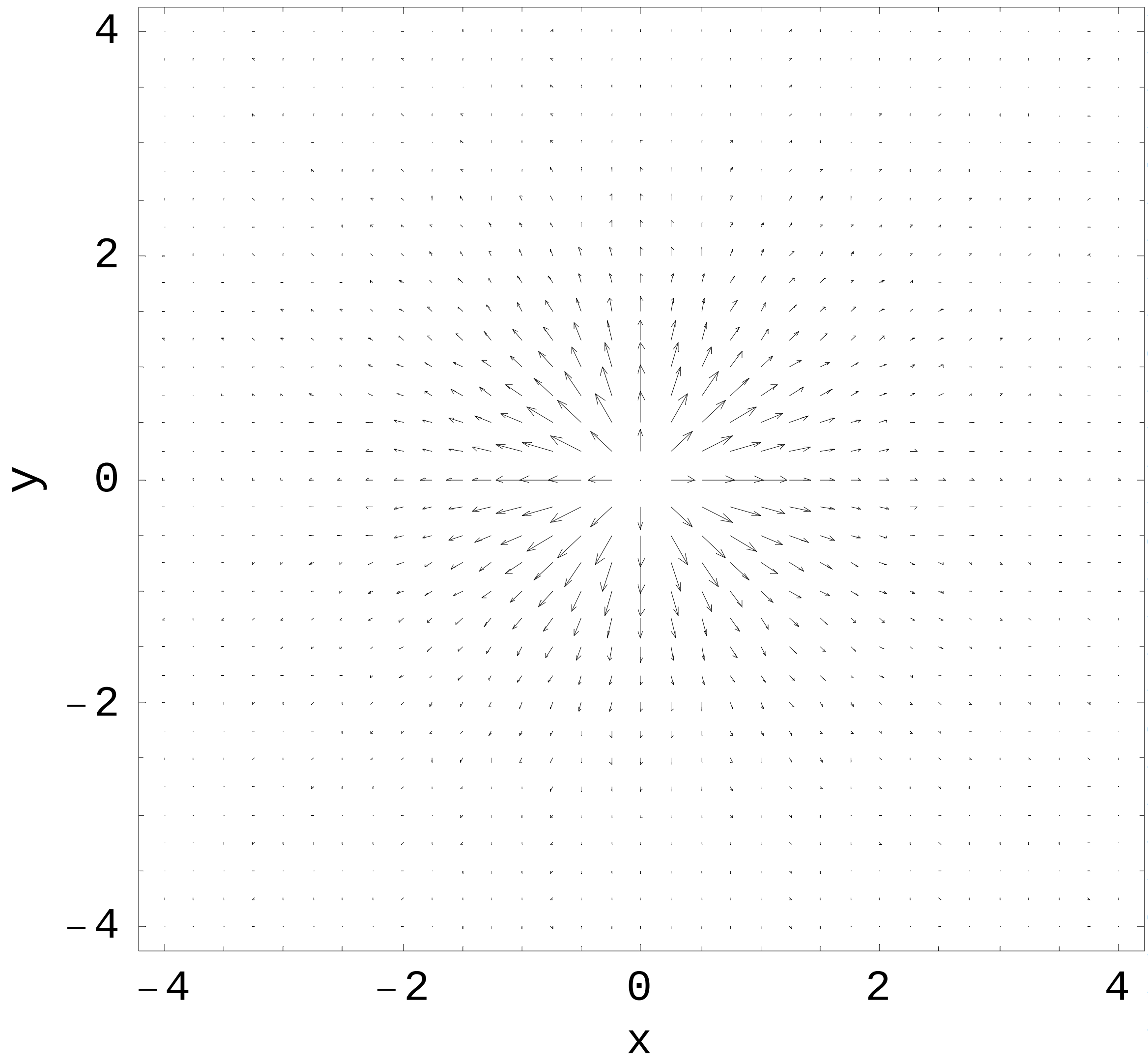


Or, in **three** dimensions as:

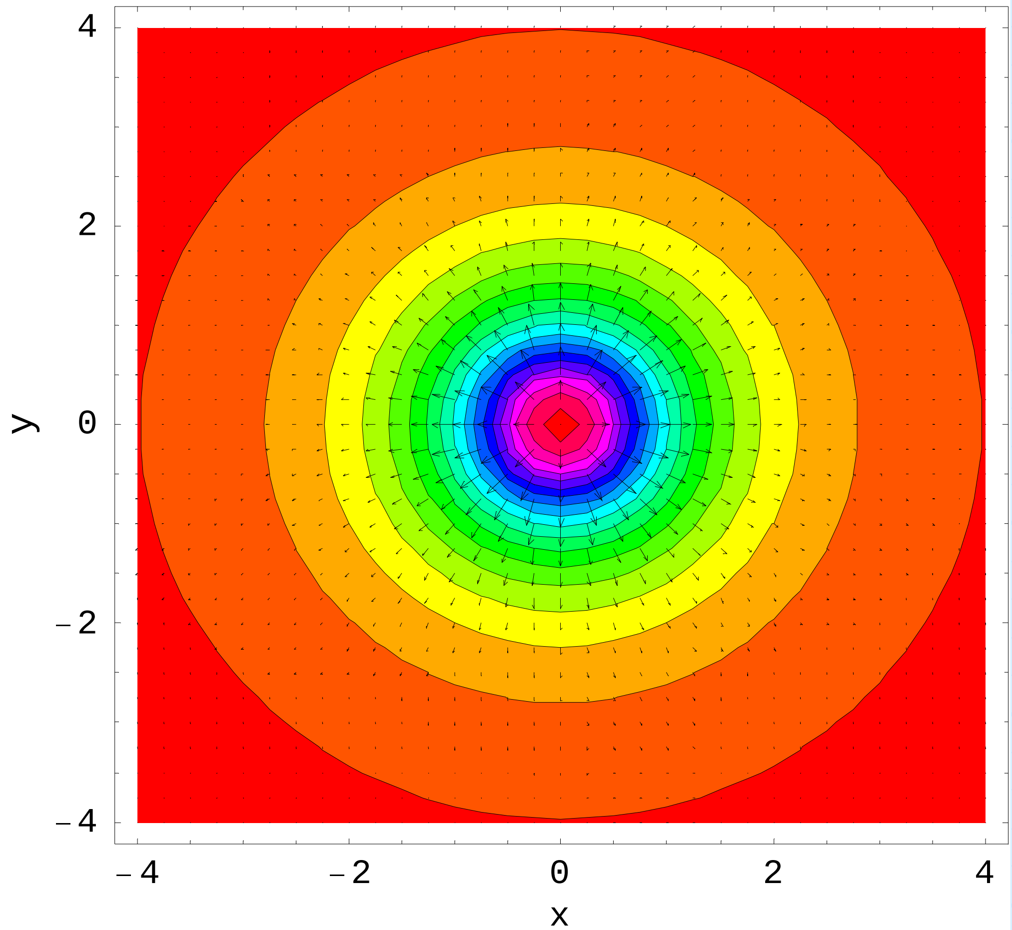


Note the electric potential **increases** as we get **closer** to the point charge (located at the origin). It appears that we have "**mountain**" of electric potential; an appropriate analogy, considering that the potential energy of a mass in the Earth's gravitational field increases with altitude (i.e., height)!

Recall the **electric field** produced by a point charge is a **vector** field that looks like:

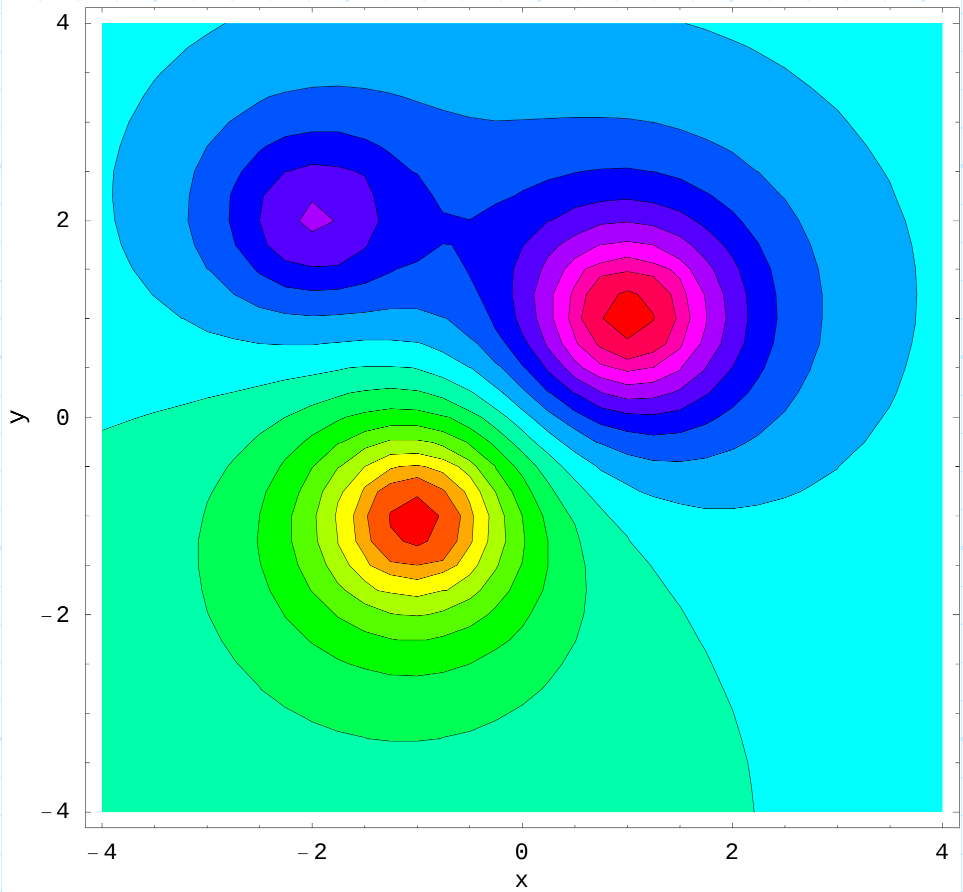


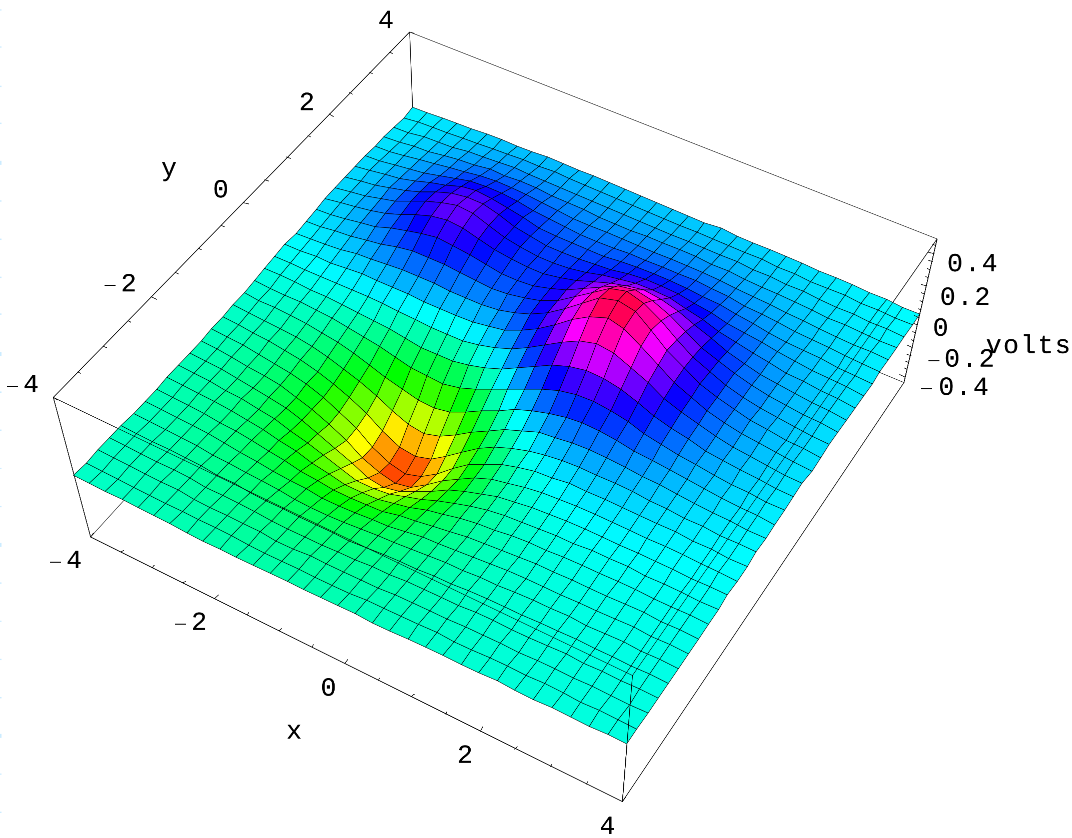
Combining the electric **field** plot with the electric **potential** plot, we get:

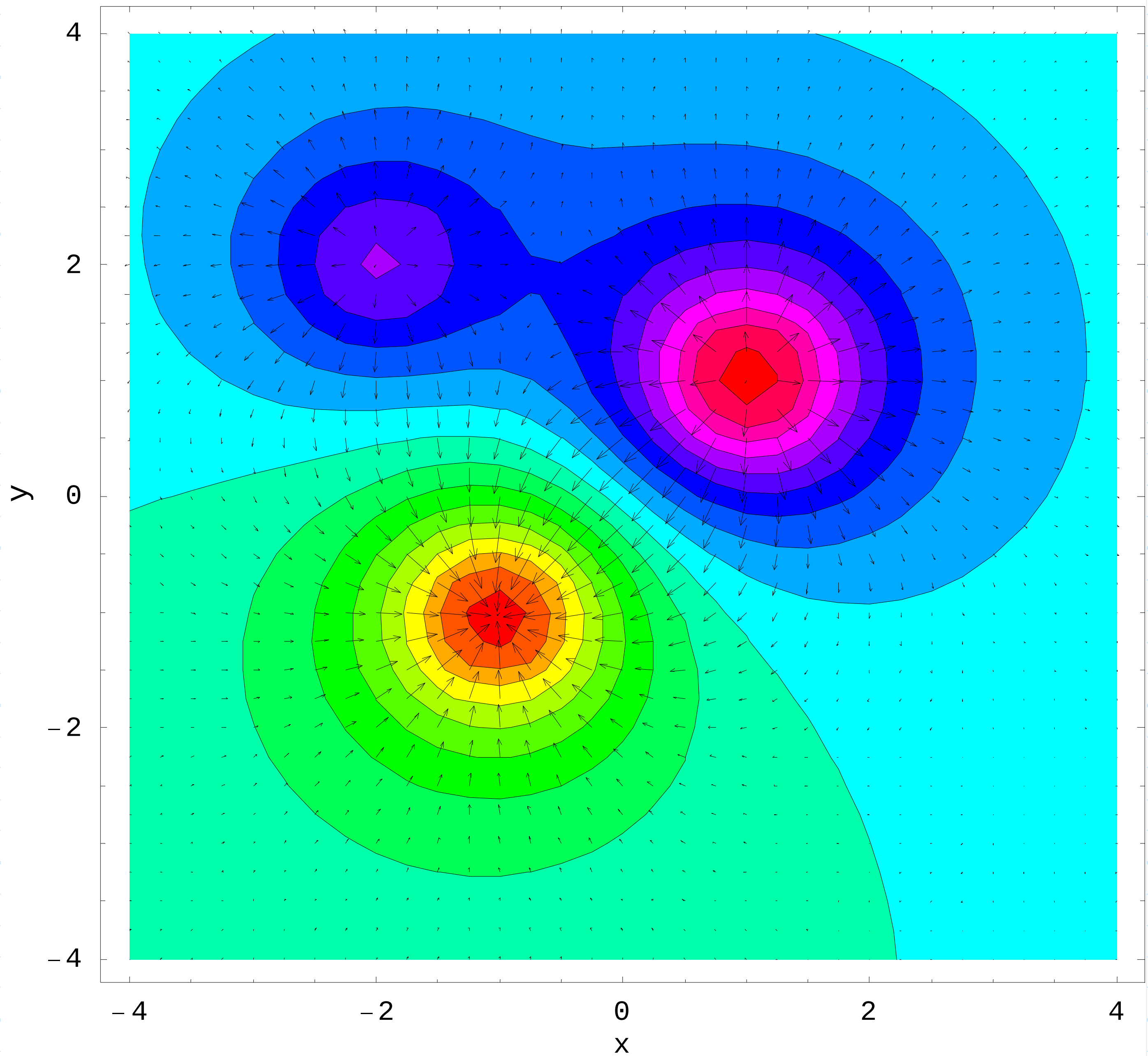


Given our understanding of the **gradient**, the above plot makes perfect sense! Do **you** see why ?

Now lets examine another **example**, where three point charges (one of them **negative!**) are present.


4
2
y 0
-2
-4
-4
-2
0
2
4
x


4
2
y
0
-2
-4
-4
-2
0
x
2
4
0.4
0.2
0
-0.2
-0.4
volts


4
2
0
−2
−4
y
−4
−2
0
2
4
x

Electric Potential Function for Charge Densities

Recall the total static electric field produced by 2 **different** charges (or charge densities) is just the **vector sum** of the fields produced by each:

$$\mathbf{E}(\bar{\mathbf{r}}) = \mathbf{E}_1(\bar{\mathbf{r}}) + \mathbf{E}_2(\bar{\mathbf{r}})$$

Since the fields are conservative, we can write this as:

$$\begin{aligned}\mathbf{E}(\bar{\mathbf{r}}) &= \mathbf{E}_1(\bar{\mathbf{r}}) + \mathbf{E}_2(\bar{\mathbf{r}}) \\ -\nabla V(\bar{\mathbf{r}}) &= -\nabla V_1(\bar{\mathbf{r}}) - \nabla V_2(\bar{\mathbf{r}}) \\ -\nabla V(\bar{\mathbf{r}}) &= -\nabla\left(V_1(\bar{\mathbf{r}}) + V_2(\bar{\mathbf{r}})\right)\end{aligned}$$

Therefore, we find,

$$V(\bar{\mathbf{r}}) = V_1(\bar{\mathbf{r}}) + V_2(\bar{\mathbf{r}})$$

In other words, **superposition** also holds for the electric potential function! The total electric potential field produced by a collection of charges is simply the **sum** of the electric potential produced by **each**.

Consider now some **distribution** of charge, $\rho_v(\bar{\mathbf{r}})$. The amount of charge dQ, contained within **small volume** dv, located at position $\bar{\mathbf{r}}'$, is:

$$dQ = \rho_v(\bar{\mathbf{r}}')\, dv'$$

The **electric potential function** produced by this charge is therefore:

$$dV(\bar{r}) = \frac{dQ}{4\pi\varepsilon_0 \left|\bar{r}-\bar{r}'\right|} = \frac{\rho_v(\bar{r}')\,dv'}{4\pi\varepsilon_0 \left|\bar{r}-\bar{r}'\right|}$$

Therefore, **integrating** across all the charge in some **volume** V, we get:

$$V(\bar{r}) = \iiint_V \frac{\rho_v(\bar{r}')}{4\pi\varepsilon_0 \left|\bar{r}-\bar{r}'\right|}\,dv'$$

Likewise, for **surface** or **line** charge density:

$$V(\bar{r}) = \iint_S \frac{\rho_s(\bar{r}')}{4\pi\varepsilon_0 \left|\bar{r}-\bar{r}'\right|}\,ds'$$

$$V(\bar{r}) = \int_C \frac{\rho_\ell(\bar{r}')}{4\pi\varepsilon_0 \left|\bar{r}-\bar{r}'\right|}\,d\ell'$$

Note that these integrations are **scalar** integrations—typically they are **easier** to evaluate than the integrations resulting from **Coulomb's Law**.

Once we find the electric potential function $V(\bar{r})$, we can **then** determine the total **electric field** by taking the gradient:

$$\mathbf{E}(\bar{r}) = -\nabla V(\bar{r})$$

Thus, we now have **three** (!) potential methods for determining the **electric field** produced by some **charge distribution** $\rho_v(\bar{r})$:

1. Determine $\mathbf{E}(\bar{r})$ from **Coulomb's Law**.

2. If $\rho_v(\bar{r})$ is symmetric, determine $\mathbf{E}(\bar{r})$ from **Gauss's Law**.

3. Determine the **electric potential function** $V(\bar{r})$, and then determine the electric field as $\mathbf{E}(\bar{r}) = -\nabla V(\bar{r})$.

Q: *Yikes! Which of the three should we use??*

A: To a certain extent, it does **not matter**! All three will provide the **same** result (although $\rho_v(\bar{r})$ **must** be symmetric to use method 2!).

However, **if** the charge density is symmetric, we will find that using Gauss's Law (method 2) will **typically** result in much less work!

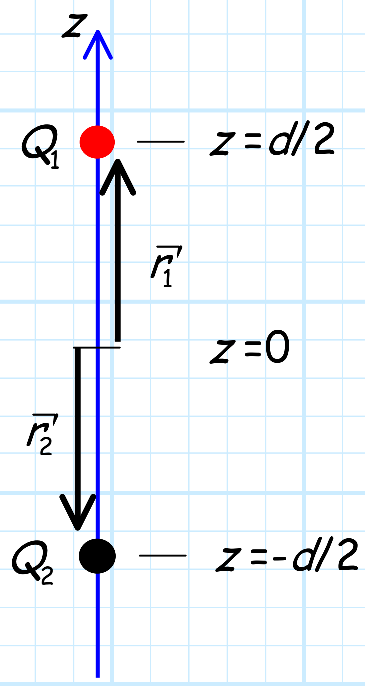
Otherwise (i.e., for **non**-symmetric $\rho_v(\bar{r})$), we find that **sometimes** method 1 is easiest, but in **other** cases method 3 is a bit less stressful (i.e., **you** decide!).

Example: The Electric Dipole

Consider **two** point charges (Q_1 and Q_2), each with **equal magnitude** but **opposite sign**, i.e.:

$$Q_1 = Q \quad \text{and} \quad Q_2 = -Q \qquad \text{so} \quad Q_1 = -Q_2$$

Say these two charges are located on the z-axis, and separated by a **distance** d.



The Electric Dipole

The **location** of charge $Q_1 = -Q$ is therefore specified by **position vector** $\bar{r}_1' = \dfrac{d}{2}\hat{a}_z$

The **location** of charge $Q_2 = -Q$ is therefore specified by **position vector** $\bar{r}_2' = \dfrac{-d}{2}\hat{a}_z$

We call this charge configuration an **electric dipole**. Note the **total charge** in a dipole is **zero** (i.e., $Q_1 + Q_2 = Q - Q = 0$). But, since the charges are located at different positions, the electric field that is created is **not** zero !

Q: *Just what **is** the electric field created by an electric dipole?*

A: One approach is to use **Coulomb's Law**, and add the resulting electric **vector** fields from each charge together.

However, let's try a different approach. Let's find the **electric potential field** resulting from an electric dipole. We can then take the gradient to find the electric field !

Note that this should be relatively **straightforward**! We already know the electric potential resulting from a **single** point charge—the electric potential resulting from two point charges is simply the **summation** of each:

$$V(\bar{r}) = V_1(\bar{r}) + V_2(\bar{r})$$

where the electric potential $V_1(\bar{r})$, created by charge Q_1, is:

$$V_1(\bar{r}) = \frac{Q_1}{4\pi\varepsilon_0 \left|\bar{r} - \bar{r}_1\right|} = \frac{Q}{4\pi\varepsilon_0 \left|\bar{r} - \frac{d}{2}\hat{a}_z\right|}$$

and electric potential $V_2(\bar{r})$, created by charge Q_2, is:

$$V_2(\bar{r}) = \frac{Q_2}{4\pi\varepsilon_0 \left|\bar{r} - \bar{r}_2\right|} = \frac{-Q}{4\pi\varepsilon_0 \left|\bar{r} + \frac{d}{2}\hat{a}_z\right|}$$

Therefore the **total** electric potential field is:

$$\begin{aligned} V(\bar{r}) &= \frac{Q}{4\pi\varepsilon_0 \left|\bar{r} - \frac{d}{2}\hat{a}_z\right|} - \frac{Q}{4\pi\varepsilon_0 \left|\bar{r} + \frac{d}{2}\hat{a}_z\right|} \\ &= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\left|\bar{r} - \frac{d}{2}\hat{a}_z\right|} - \frac{1}{\left|\bar{r} + \frac{d}{2}\hat{a}_z\right|}\right) \end{aligned}$$

If the point denoted by $\bar{r}$ is a significant distance away from the electric dipole (i.e., $|\bar{r}| >> d$), we can use the following **approximations**:

$$\frac{1}{\left|\bar{r} - \frac{d}{2}\hat{a}_z\right|} \approx \frac{1}{|\bar{r}|} + \frac{d\cos\theta}{2|\bar{r}|} = \frac{1}{r} + \frac{d\cos\theta}{2r^2}$$

$$\frac{1}{\left|\bar{r} + \frac{d}{2}\hat{a}_z\right|} \approx \frac{1}{|\bar{r}|} - \frac{d\cos\theta}{2|\bar{r}|} = \frac{1}{r} - \frac{d\cos\theta}{2r^2}$$

where r and θ are the **spherical coordinate** variables of the point denoted by $\bar{r}$.

Therefore, we find:

$$\begin{aligned} V(\bar{r}) &= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\left|\bar{r}-\frac{d}{2}\hat{a}_z\right|}-\frac{1}{\left|\bar{r}+\frac{d}{2}\hat{a}_z\right|}\right) \\ &= \frac{Q}{4\pi\varepsilon_0}\left(\left(\frac{1}{r}+\frac{d\cos\theta}{2r^2}\right)-\left(\frac{1}{r}-\frac{d\cos\theta}{2r^2}\right)\right) \\ &= \frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2} \end{aligned}$$

Note the result. The **electric potential field** produced by an **electric dipole,** when centered at the **origin** and aligned with the ***z*-axis** is:

$$V(\bar{r}) = \frac{Qd}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}$$

Q: *But the original question was, what is the **electric field** produced by an electric dipole?*

A: Easily determined! Just take the **gradient** of the electric potential function, and multiply by -1.

$$
\begin{aligned}
\mathbf{E}(\bar{\mathrm{r}}) &= -\nabla V(\bar{\mathrm{r}}) \\
&= -\nabla\left(\frac{Qd}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}\right) \\
&= \frac{-Q\,d}{4\pi\varepsilon_0}\left[\cos\theta\;\frac{d}{dr}\left(\frac{1}{r^2}\right)\hat{a}_r + \frac{1}{r^3}\frac{d(\cos\theta)}{d\theta}\hat{a}_\theta\right] \\
&= \frac{-Q\,d}{4\pi\varepsilon_0}\left[\left(\frac{-2\cos\theta}{r^3}\right)\hat{a}_r - \frac{\sin\theta}{r^3}\hat{a}_\theta\right]
\end{aligned}
$$

The static **electric field** produced by an **electric dipole**, when centered at the **origin** and aligned with the ***z*-axis** is:

$$
\mathbf{E}(\bar{\mathrm{r}}) = \frac{Qd}{4\pi\varepsilon_0}\frac{1}{r^3}\left[\;2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right]
$$

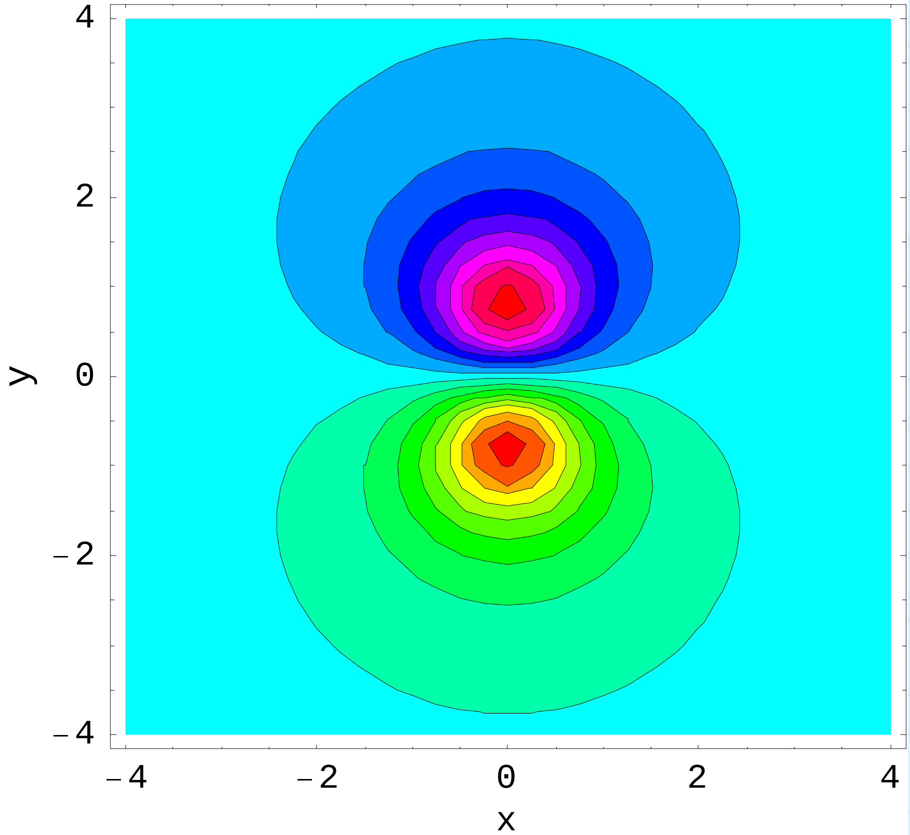
Yikes! **Contrast** this with the electric field of a **single** point charge. The electric dipole produces an electric field that:

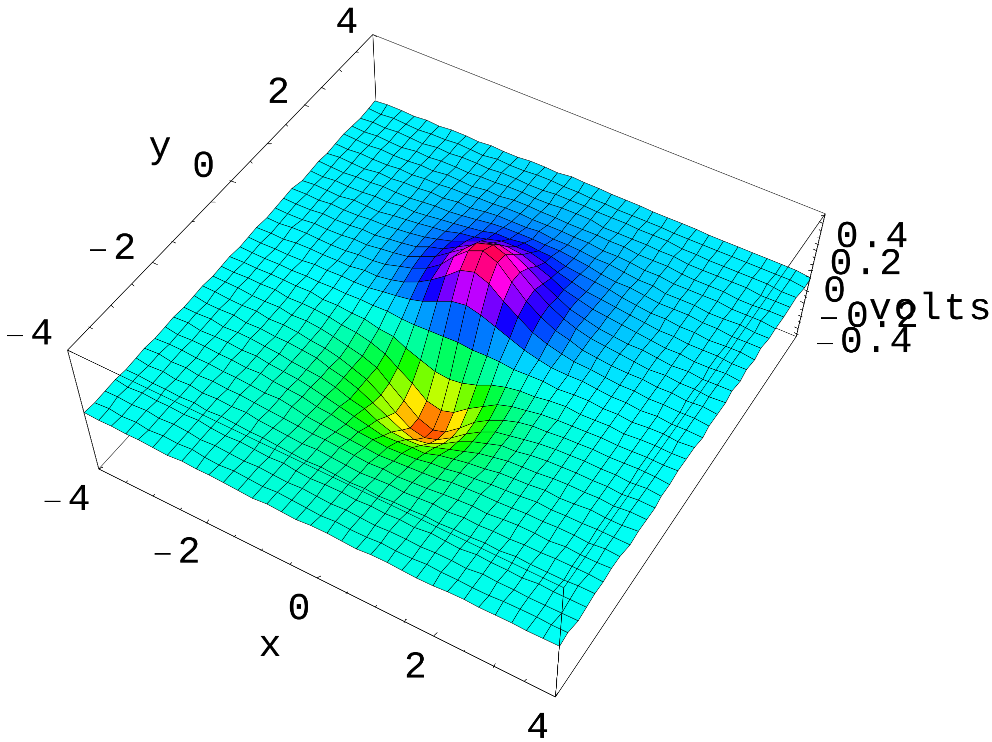
1) Is proportional to r^{-3} (as opposed to r^{-2}).

2) Has vector components in **both** the $\hat{a}_r$ and $\hat{a}_\theta$ directions (as opposed to just $\hat{a}_r$).

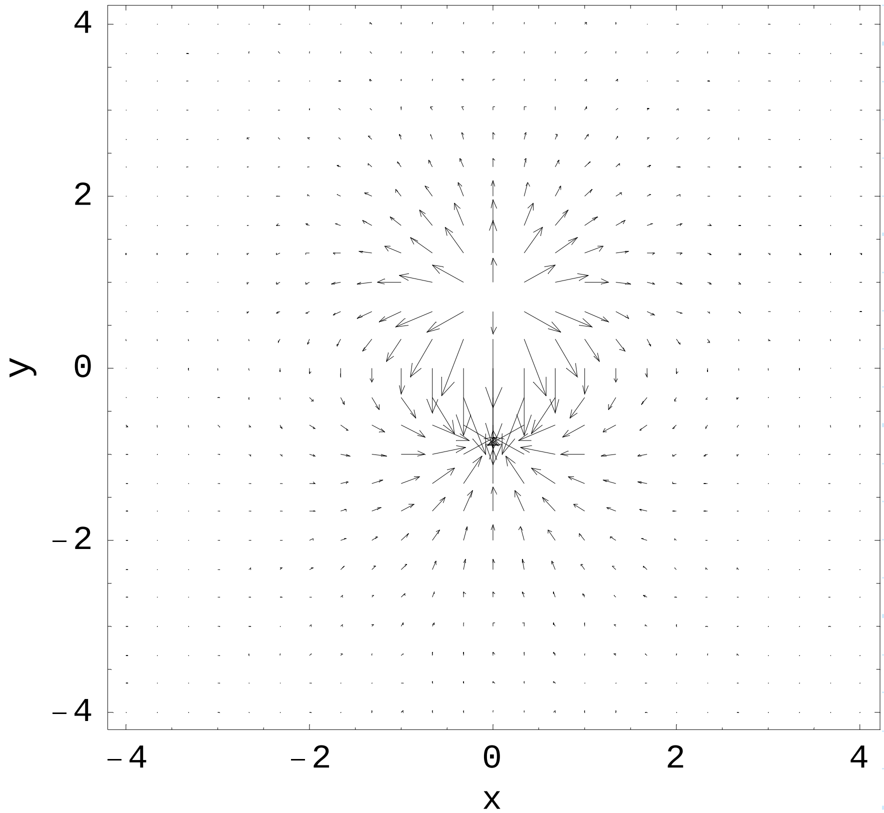
In other words, the electric field does **not** point away from the electric dipole!

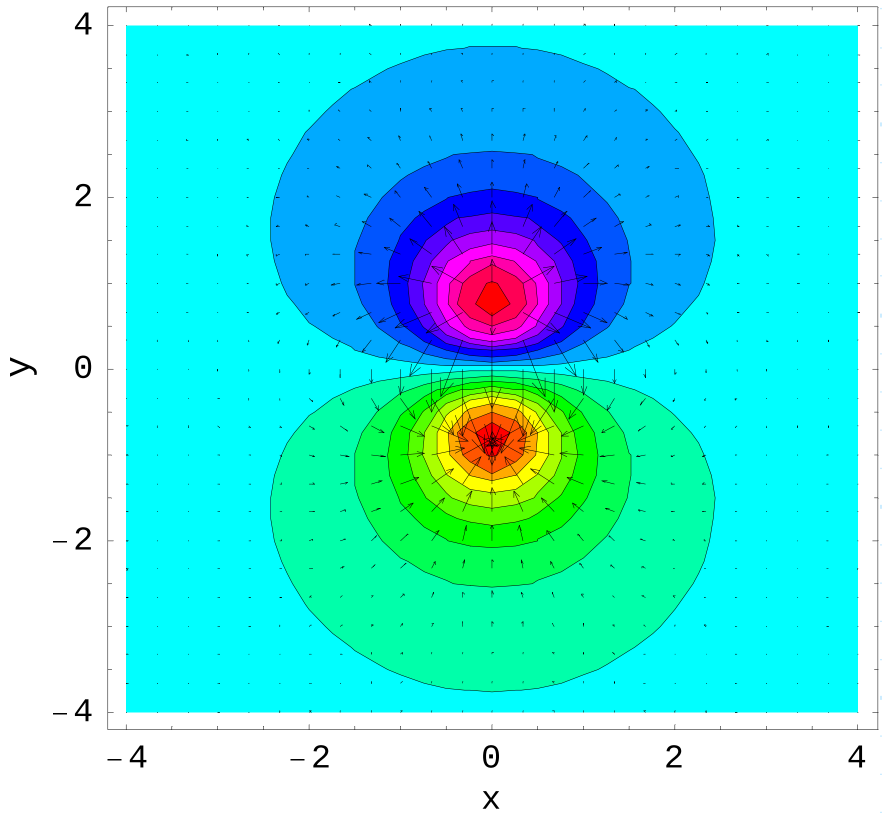
The **electric potential** produced by an electric dipole looks like:





And the **electric field** produced by the electric dipole is:





The Dipole Moment

Note that the dipole solutions:

$$V(\bar{\mathrm{r}}) = \frac{Qd}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}$$

and

$$\mathbf{E}(\bar{\mathrm{r}}) = \frac{Qd}{4\pi\varepsilon_0}\frac{1}{r^3}\left[\ 2\cos\theta\ \hat{a}_r + \sin\theta\ \hat{a}_\theta\ \right]$$

provide the fields produced by an electric dipole that is:

1. **Centered** at the origin.

2. **Aligned** with the *z*-axis.

Q: *Well isn't that just grand. I suppose these equations are thus **completely useless** if the dipole is **not** centered at the origin and/or is **not** aligned with the z-axis !*!@!*

A: That is indeed **correct**! The expressions above are **only** valid for a dipole centered at the origin and aligned with the *z*-axis.

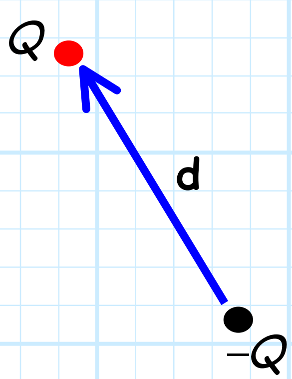
To determine the fields produced by a more **general case** (i.e., arbitrary location and alignment), we first need to **define** a new quantity **p**, called the **dipole moment**:

$$\mathbf{p} = Q\,\mathbf{d}$$

Note the dipole moment is a **vector** quantity, as the **d** is a vector quantity.

Q: *But what the heck is vector **d** ??*

A: Vector **d** is a **directed distance** that extends **from** the location of the **negative** charge, **to** the location of the **positive** charge. This directed distance vector **d** thus describes the **distance** between the dipole charges (vector magnitude), as well as the **orientation** of the charges (vector direction).



Therefore $\mathbf{d} = |\mathbf{d}|\,\hat{a}_d$, where:

$|\mathbf{d}|$ = **distance** d between charges

and

$\hat{a}_d$ = the **orientation** of the dipole

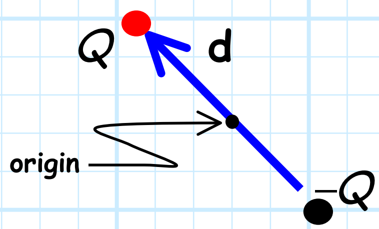
Note **if** the dipole is aligned with the *z*-axis, we find that $\mathbf{d} = d\ \hat{a}_z$. Thus, since $\hat{a}_z \cdot \hat{a}_r = \cos\theta$, we can write the expression:

$$\begin{aligned} Qd\ \cos\theta &= Q\ d\ \hat{a}_z \cdot \hat{a}_r \\ &= Q\ \mathbf{d} \cdot \hat{a}_r \\ &= \mathbf{p} \cdot \hat{a}_r \end{aligned}$$

Therefore, the electric potential field created by a dipole centered at the origin and aligned with the *z*-axis can be rewritten in terms of its dipole moment **p**:

$$\begin{aligned} V(\bar{\mathrm{r}}) &= \frac{Qd}{4\pi\varepsilon_0} \frac{\cos\theta}{r^2} \\ &= \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p} \cdot \hat{a}_r}{r^2} \end{aligned}$$

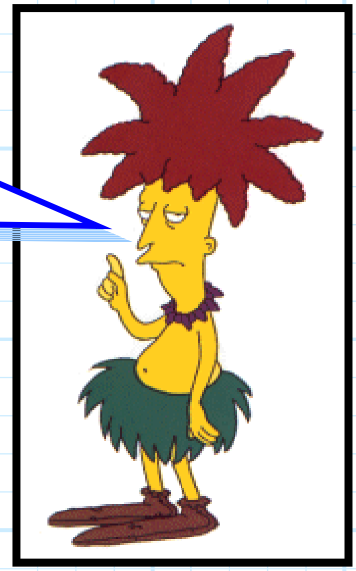
It turns out that, not **only** is this representation valid for a dipole aligned with the *z*-axis (e.g., $\mathbf{d} = d\,\hat{a}_z$), it is valid for electric dipoles located at the origin, and oriented in **any** direction!



$$V(\bar{\mathrm{r}}) = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p} \cdot \hat{a}_r}{r^2}$$

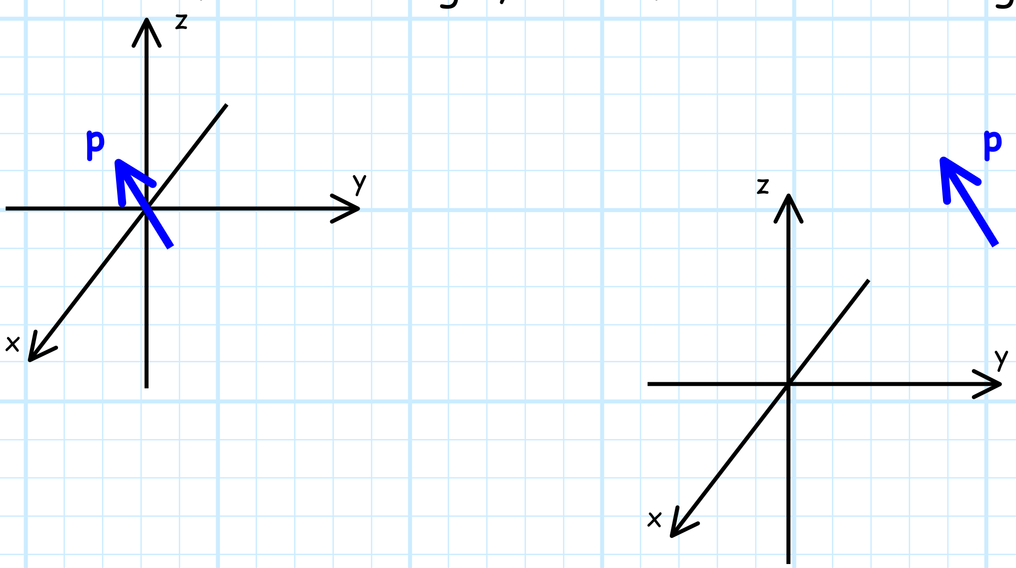
Although the expression above is valid for **any** and **all** dipole moments **p**, it is valid **only** for dipoles located at the origin (i.e., $\bar{\mathrm{r}}' = 0$).

Q: ***Swell.*** *But you have* ***neglected*** *one significant detail—what are the fields produced by a dipole when it is* ***NOT*** *located at the origin?*



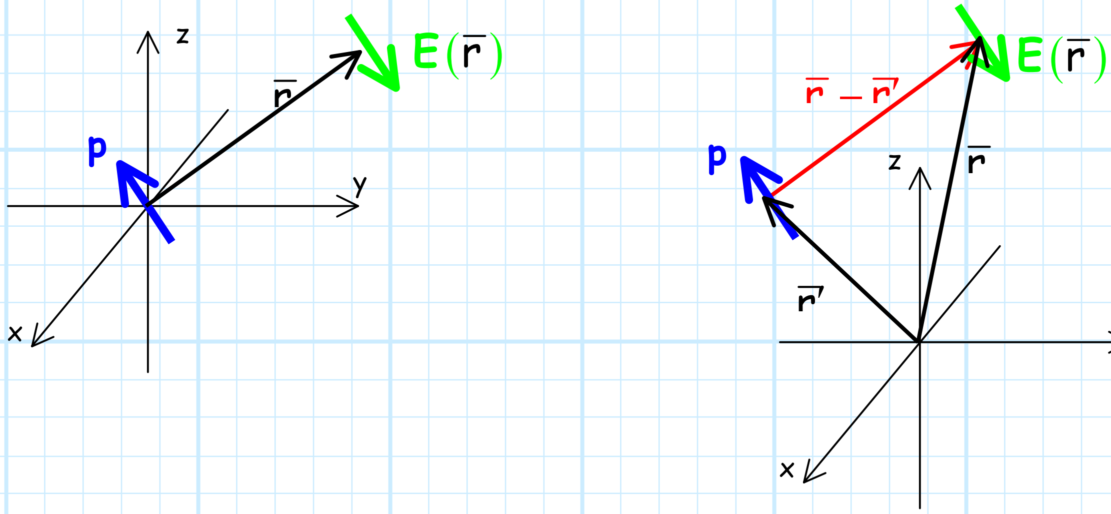
A: Finding the solution for **this** problem is our next task!

Note the electric dipole does **not** "know" where the origin is, or if it is located there. As far as the **dipole** is concerned, we do not move it from the origin, but in fact move the origin from **it**!



In other words, the fields produced by an electric dipole are **independent** of its location or orientation—it is the mathematics expressing these fields that get modified when we change our origin and coordinate system!

Thus, we simply need to **translate** the previous field (dipole at the origin) solution by the **same** distance and direction that we move the dipole from the origin.



Just as with charge, the **location** of the dipole (center) is denoted by position vector $\bar{r}'$.

Note if the dipole is located at the origin, the position vector $\bar{r}$ extends from the dipole the location where we evaluate the electric field.

However, if the dipole is **not** located at the origin, this vector extending from the dipole to the electric field is **instead** $\bar{r}-\bar{r}'$. Thus, to translate the solution of the dipole at the origin to a new location, we replace vector $\bar{r}$ with vector $\bar{r}-\bar{r}'$, i.e.:

$$r = |\bar{\mathbf{r}}| \qquad \text{becomes} \qquad |\bar{\mathbf{r}} - \bar{\mathbf{r}}'|$$

$$\hat{a}_r = \frac{\bar{\mathbf{r}}}{|\bar{\mathbf{r}}|} \qquad \text{becomes} \qquad \hat{a}_R = \frac{\bar{\mathbf{r}} - \bar{\mathbf{r}}'}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|}$$

$$V(\bar{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{a}_r}{r^2} \qquad \text{becomes} \qquad V(\bar{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot(\bar{\mathbf{r}}-\bar{\mathbf{r}}')}{|\bar{\mathbf{r}}-\bar{\mathbf{r}}'|^3}$$

Thus, a dipole of **any** arbitrary orientation and location produces the electric potential field:

$$\begin{aligned} V(\bar{\mathbf{r}}) &= \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{a}_R}{|\bar{\mathbf{r}}-\bar{\mathbf{r}}'|^2} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot(\bar{\mathbf{r}}-\bar{\mathbf{r}}')}{|\bar{\mathbf{r}}-\bar{\mathbf{r}}'|^3} \end{aligned}$$